CENTRAL LIMIT THEOREM FOR
PARKING MODELS ON THE LINE AND PLANE

BY

HOWARD J. WEINER

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by Howard J. Weiner

University of California at Davis and Stanford University*

1.   Introduction. Three models for random parking ($\equiv$ packing) of line

segments on a longer line and correspondingly, squares on a larger

rectangle are given, where for each model, either the Renyi or the

Solomon parking method is used.

The one-dimensional first moment results are given in Solomon (1966),

(1970), Weiner (1978).

In the alternating car size model, a car of length a is placed

uniformly at random along a curb (segment) of length x. It is considered

praked there. A second car, of length b is placed uniformly at random

along the curb and parked if and only if it does not overlap the a-car.

This is the Renyi parking mechanism. In the Solomon parking mechanism,

a newly-placed car which overlaps an already-parked car is moved the

shorter distance so as to be immediately adjacent to the overlapped car

on the curb, and is parked there if and only if it does not now overlap

yet another parked car, or the boundary of the segment. If a b-car is not

parked, it is discarded, and another b-car is placed I.I.D. at random as

the other b-car. If it is parked, then an a-car is placed, and the process

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continues until no further cars can be parked. If the b-car is not parked, again it is discarded and another b-car is placed, and again the process continues until no more cars may be parked. This process may be similarly defined for the plane for $a \times a$ and $b \times b$ squares, respectively, with either of the Renyi and Solomon mechanisms.

The abacus model consists of a rectangle, $(k+1) \times x$, of k horizontal lines, each one unit apart. The width of the rectangle (the length of each line) is x. Assume a, b are integers. A line is chosen uniformly at random from the integers $\{1,2,\ldots,k\}$, and centered on this line, an $a \times a$ (for definiteness) square is placed. Next, a $b \times b$ square is selected and placed uniformly on a line, and parked if there is no overlap. If there is overlap with the $a \times a$ parked car on a given line (Renyi model) or if a car to be placed is vertically adjacent to some portion of a car, without a line of space between cars, the $b \times b$ car is placed I.I.D. as before until it is parked. Then another $a \times a$ car is placed I.I.D. and the process continues until no further cars may be parked. The Solomon model allows that if two cars overlap on the same line, the one to be placed may be moved adjacent horizontally (minimal motion) to the already-parked car and parked if there is now no overlap with another parked car, if the boundary is not overlapped, and if at least one line of vertical space exists between the car and already-parked cars on other lines.

The random car size model to be considered has $D \times D$ square cars to be parked, either on a rectangle or abacus with the Renyi mechanism, where D is chosen from a distribution F, with density $f(x) > 0$, all $x > 0$. If a $D \times D$ car fits the boundaries of the given rectangle and does not overlap another already-parked car, (and leaves enough vertical space in the abacus

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grid case), it is parked, otherwise it is discarded and the process stops. If the car is parked, another D is chosen I.I.D. from F and a $D \times D$ car is placed I.I.D. as the other cars, and if it is not parked as described above, the process stops.

2. <u>One-Dimensional Models</u>. Define, for $a, b \ll x$,

(2.1) $X_a(x)$ $(X_b(x))$ = the random variable of total number of cars which may be parked on a curb of length x in the alternating a and b length car Solomon model, starting with an a-car (b-car)

$$K_a(x) \equiv E(X_a(x))$$
$$K_b(x) \equiv E(X_b(x)).$$

By considerations as in (Solomon (1966)), it may be shown that

(2.2) $$\lim_{x \to \infty} x^{-1} \begin{pmatrix} K_a(x) \\ K_b(x) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \underline{\beta} .$$

Define

(2.3) $$L_a(s) = \int_0^\infty e^{-sx} K_a(x) \, dx$$

$$L_b(s) = \int_0^\infty e^{-sx} K_b(x) \, dx$$

that for some constants β_1, β_2, δ_1, δ_2 which can be determined, for $s \downarrow 0$

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(2.4)
$$L_a(s) \sim \beta_1 s^{-2} + \delta_1 s^{-1}$$

$$L_b(s) \sim \beta_2 s^{-2} + \delta_2 s^{-1} \ .$$

Let

(2.5)
$$\ell_a(x) \equiv \beta_1 x + \delta_1$$

$$\ell_b(x) \equiv \beta_2 x + \delta_2 \ .$$

It may be shown that $\underline{\ell}(x)$ satisfies the same integral equation as $\underline{K}(x) \equiv \begin{pmatrix} K_a(x) \\ K_b(x) \end{pmatrix}$. Hence

(2.6) $\quad X_a(x+a) - \ell_a(x+a) = (X_b(x) - \ell_b(x)) \chi_{[(-a,0) \cup (x,x+a)]}$

$$+ [(X_b'(t) - \ell_b(t)) + (X_b''(x-t) - \ell_b(x-t))]$$

and a similar equation for $X_b(x+b) - \ell_b(x+b)$, where the a, b are inter-changed, where t is chosen uniformly on $[0,x]$, $X_a'(t)$, $X_a''(t)$ are I.I.D. as $X_a(t)$, conditional on t. Squaring both sides of (2.6) and taking expectations, with $M_a^{(2)}(x) \equiv E(X_a(x) - \ell_a(x))^2$,

(2.7) $\quad M_a^{(2)}(x+a) = \dfrac{2a}{x+2a} M_b^{(2)}(x) + \dfrac{2}{x+2a} \displaystyle\int_0^x M_b^{(2)}(t)\, dt$

$$+ \dfrac{2}{x+2a} \int_0^x m_b(t) m_b(x-t)\, dt,$$

and

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$$M_b^{(2)}(x+b) = \frac{2b}{x+2b} M_a^{(2)}(x) + \frac{2}{x+2b} \int_0^x M_a^{(2)}(t)\, dt + \frac{2}{x+2b} \int_0^x m_a(t) m_a(x-t)\, dt,$$

where

(2.8)
$$m_a(t) \equiv E(X_a(t) - \ell_a(t)),$$

$$m_b(t) \equiv E(X_b(t) - \ell_b(t)).$$

Define

(2.9)
$$L_a(s) \equiv \int_a^\infty e^{-sx} M_a^{(2)}(x)\, dx$$

$$L_b(s) \equiv \int_b^\infty e^{-sx} M_b^{(2)}(x)\, dx$$

$$r_a(s) \equiv \int_a^\infty e^{-sx} m_a(t)\, dx$$

$$r_b(s) \equiv \int_b^\infty e^{-sb} m_b(x)\, dx$$

$$\underline{L} = \begin{pmatrix} L_a(s) \\ L_b(s) \end{pmatrix}$$

Taking Laplace transforms in (2.7), one obtains

(2.10)
$$\underline{L}' + A\underline{L} = -\underline{B}$$

where

$$A = \begin{pmatrix} -a, & 2ae^{-sa} + \dfrac{2e^{-as}}{s} \\[2ex] 2be^{-sb} + \dfrac{2e^{-sb}}{s}, & -b \end{pmatrix}$$

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$$B = \begin{pmatrix} 2e^{-sa}r_b^2 + \dfrac{(2ae^{-as}-3ae^{-2as})}{s} + \dfrac{(e^{-as}-e^{-2as})}{s^2} \\[3mm] 2e^{-sb}r_a^2 + \dfrac{(2be^{-bs}-3be^{-2bs})}{s} + \dfrac{(e^{-bs}-e^{-2bs})}{s^2} \end{pmatrix} .$$

Define

(2.11) $K_{1a}(x)(K_{1b}(x))$ = mean total number of a-length (b-length)

cars which can be parked on $[0,x)$.

Assume $a < b$.

Clearly

(2.12) $K_{1b}(x) \le K_a(x), K_b(x) \le K_{1a}(x)$,

and a similar inequality holds for the respective non-normalized second

moments.

Hence we seek a solution, either by power series, or Volterra-like

integral equation representation to (2.10) such that

(2.13) $\lim_{s \downarrow 0} s^2(L_i(s) = \alpha_i > 0,$ $i=1,2.$

Hence $M_a^{(2)}(x), M_b^{(2)}(x)$ are respectively expressible as a sum of three

terms, each of which is increasing in absolute value. By Abelian, Tauberian

theorems (Widder (1946), p. 182 and p. 192),

(2.14) $\lim_{x \to \infty} x^{-1}\begin{pmatrix} M_a^{(2)}(x) \\ M_b^{(2)}(x) \end{pmatrix} = \lim_{x \to \infty} x^{-1}\begin{pmatrix} \text{Var } X_a(x) \\ \text{Var } X_b(x) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

For the abacus, define

(2.15) $Y_a(k)$ = total number of alternately-placed a and b segments
on a k-line one-dimensional abacus starting with an a
segment.

$Y_b(k)$ is the similar quantity starting with a b segment.

One may conclude by similar considerations as in (Solomon (1966))
that

(2.16) $\begin{pmatrix} E\ Y_a(k) \\ E\ Y_b(k) \end{pmatrix} \sim \begin{pmatrix} \ell_a(k) \\ \ell_b(k) \end{pmatrix} = \underline{\ell}(k)$

for large k, where $\ell_a(k)$, $\ell_b(k)$ are linear functions of k.

Denoting

(2.17) $M_a^{(2)}(k) = E(Y_a(k) - \ell_a(k))^2$

$M_b^{(2)}(k) = E(Y_b(k) - \ell_b(k))^2$

$m_a(k) = E(Y_a(k) - \ell_a(k))$

$m_b(k) = E(Y_b(k) - \ell_b(k))$

it follows that with k an integer, $k \gg a$ or b.

(2.18) $M_a^{(2)}(k+a) = \dfrac{2}{k} \sum_{\ell=1}^{k} m_b(\ell)m_b(k-\ell) + \dfrac{2}{k} \sum_{\ell=1}^{k} M_b^{(2)}(\ell)$

$M_b^{(2)}(k+b) = \dfrac{2}{k} \sum_{\ell=1}^{k} m_a(\ell)m_a(k-\ell) + \dfrac{2}{k} \sum_{\ell=1}^{k} M_a^{(2)}(\ell).$

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Similar considerations to those above yield

$$(2.19) \qquad \lim_{k \to \infty} k^{-1} \binom{\text{Var } Y_a(k)}{\text{Var } Y_b(k)} = \underline{\eta} > 0.$$

For the random car size model in one dimension, define

(2.20)   Z(x) = total number of cars of length D, where D has

distribution F, density f, which may be parked

on a curb of length x .

Define

$$(2.21) \qquad M(x) = E \; Z(x)$$

$$L(s) = \int_0^\infty e^{-sx} M(x) \; dx$$

$$\varphi(s) = \int_0^\infty e^{-sx} f(x) \; dx.$$

It may be shown with the prior model that for $s \downarrow 0$, a ν, λ may be computed which satisfy, for $s \downarrow 0$,

(2.22)
$$L(s) \sim \nu s^{-2} + \lambda s^{-1}.$$

Define, for $x > 0$

(2.23)
$$\ell(x) \equiv \nu x + \lambda$$
$$m(x) = = M(x) - \ell(x)$$
$$M^{(2)}(x) = E(Z(x) - \ell(x))^2$$

By the methods of earlier paragraphs, it follows that, for $a > 0$,

(2.24)
$$M^{(2)}(a) = \frac{2}{a}\left[\int_0^a f(x) \, dx \int_0^{a-x} M^{(2)}(t) \, dt \right.$$
$$\left. + \int_0^a f(x) \, dx \int_0^{a-x} m(t)m(a-x-t) \, dt \right].$$

If $E(D^2) < \infty$, it follows that

(2.25)
$$\lim_{x \to \infty} x^{-1} M^{(2)}(x) = \beta > 0.$$

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3. <u>Higher Moments</u>.

    Let

(3.1) $X(x)$ = total number of unit length segments (cars) which may

be parked on $[0,x)$ in accord with a Renyi model, where each unit segment

is placed uniformly at random and parked only if it does not overlap a

previously parked car. See (Solomon (1966)).

    It may be shown (Solomon (1966)) that for $x \to \infty$

(3.2) $E X(x) \equiv m(x) \sim \ell(x) \equiv cx + c - 1$

where $c \sim .748$.

    Define, for $n \geq 1$

(3.3) $M^{(n)}(x) \equiv E(X(x) - \ell(x))^n ,$

and $M^{(1)}(x) \equiv m(x)$

    It is known that (Solomon (1966))

(3.4) $\lim_{x \to \infty} x^{-1} M^{(2)}(x) = \lim_{x \to \infty} x^{-1} \mathrm{Var}\, X(x) = \alpha > 0.$

    Define

(3.5) $X'(x), X''(x)$ to be I.I.D. as $X(x)$.

<u>Theorem 1</u>.

(3.6)
$$\lim_{x \to \infty} x^{-n} M^{(2n)}(x) = \frac{(2n)!}{2^n n!} \alpha^{2n}$$

and
$$\lim_{x \to \infty} x^{-(2n+1)/2} M^{(2n+1)}(x) = 0$$

<u>Proof</u>. Conditional on the placement of the first unit car on $(t, t+1)$, $x > t + 1 > t > 0$, considerations of (Weiner (1978)) yield that for asymptotic results,

(3.7) $\quad E(X(x+1) - \ell(x+1))^n \approx E([X'(t) - \ell(t)] + [X''(x-t) - \ell(x-t)])^n,$

where t is chosen uniformly on $[0,x]$. Expanding the right side by the binomial expansion and taking expectations for $n = 3$ yields

(3.8) $\quad M^{(3)}(x+1) = 6x^{-1} \int_0^t M^{(2)}(t) m(x-t) + \frac{2}{x} \int_0^x M^{(3)}(t) \, dt$

and taking Laplace transforms, with

(3.9)
$$L^{(n)}(s) = e^s \int_0^\infty e^{-sx} M^{(n)}(x) \, dx,$$

$$R^{(n)}(s) = \int_0^\infty e^{-sx} M^{(n)}(x) \, dx,$$

it follows that

(3.10) $\quad (L^{(3)}(s))' + \frac{2e^{-s}}{s} L^{(3)}(s) = -6R^{(2)}(s) R^{(1)}(s)$

with solution

$$(3.11) \quad L^{(3)}(s) = \left[\exp\left(2\int_s^\infty \frac{e^{-u}}{u}\,du\right)\right]\left[\int_s^\infty \left[\exp\left(-2\int_v^\infty \frac{e^{-u}}{u}\,du\right)\right]\left[6R^{(2)}(v)R^{(1)}(v)\right]dv\right].$$

Since for $0 < x < 1$,

$$(3.12) \qquad \exp\left(2\int_s^\infty \frac{e^{-u}}{u}\,du\right) = -2\gamma - 2\ln s + g(s),$$

where γ = Euler's constant and $g(s) \to 0$ as $s \to 0$,

$$(3.13) \qquad \lim_{s \downarrow 0} s^2 R^{(2)}(s) = \alpha^2$$

$$\lim_{s \downarrow 0} s^2 R^{(1)}(s) = 0,$$

it follows that

$$(3.14) \quad \lim_{s \downarrow 0} s^2 L^{(3)}(s) = \int_0^\infty \left[\exp\left(-2\int_v^\infty \frac{e^{-u}}{u}\,du - 2\gamma\right)\right]\left[6R^{(2)}(v)R^{(1)}(v)\right]dv = \text{constant.}$$

For $n = 4$,

$$(3.15) \qquad M^{(4)}(x+1) = \binom{4}{2}x^{-1}\int_0^x M^{(2)}(t)M^{(2)}(x-t)\,dt + q(x)$$

$$+ \frac{2}{x}\int_0^x M^{(4)}(t)\,dt$$

where

$$(3.16) \quad q(x) \text{ is of lower order of } x \text{ than } x^{-1}\int_0^x M^{(2)}(t)M^{(2)}(x-t)\,dt$$

by the results for $n=1,2,3$.

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Again taking Laplace transforms, the resulting equation for $L^{(4)}(s)$ has solution

$$(3.17) \quad L^{(4)}(s) = \left[\exp\left(2\int_s^\infty \frac{e^{-u}}{u}\,du\right)\right]\int_s^\infty \left[\exp\left(-2\int_v^\infty \frac{e^{-u}}{u}\,du\right)\right]\left[6(R^{(2)}(v))^2 + t(v)\right]\,dv$$

where $t(v)$ is the Laplace transform of $q(x)$.

Since

$$(3.18) \qquad \lim_{v \downarrow 0} v^2 R^{(2)}(v) = \alpha^2$$

$$\lim_{v \downarrow 0} v^4 t(v) = 0,$$

if one denotes

$$(3.19) \qquad f(s) = \int_s^\infty \left[\exp\left(-2\int_v^\infty \frac{e^{-u}}{u}\,du\right)\right]\left[6(R^{(2)}(v))^2 + t(v)\right]\,dv,$$

it follows by (3.18) and L'Hospital's rule that

$$(3.20) \qquad \lim_{s \downarrow 0} sf(s) = \left.\frac{f'(s)}{(\frac{1}{s})'}\right|_{s=0} = 6\alpha^4 e^{2\gamma}.$$

By (3.18), (3.20)

$$(3.21) \qquad \lim_{s \downarrow 0} s^3 L^{(4)}(s) = 6\alpha^4.$$

Since $M^{(4)}(x)$ is expressible as a sum of terms each increasing in absolute value, a Tauberian theorem (Widder (1946)) yields

$$(3.22) \qquad \lim_{x \to \infty} x^{-2} M^{(4)}(x) = 3\alpha^4.$$

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Assume the result of the theorem for $r=1,2,\ldots,2n$th moments by induction. By the binomial expansion,

$$(3.23) \qquad M^{(2n+1)}(x+1) = \frac{1}{x} \int_0^x \sum_{\ell=1}^{2n} \binom{2n+1}{\ell} M^{(\ell)}(t) M^{(2n+1-\ell)}(x-t)\, dt$$

$$+ \frac{2}{x} \int_0^x M^{(2n+1)}(t)\, dt.$$

In the first integral on the right side, if $M^{(\ell)}(t)$ is an even moment, and the induction hypothesis yields that

$$(3.24) \qquad M^{(\ell)}(t) M^{(2n+1-\ell)}(x-t) \leq O(t^{\ell/2}(x-t)^{2n-\ell/2}) \leq O(x^n),$$

and when Laplace transforms are taken in (3.19), then a simple computation by L'Hospital's rule, and an Abelian and Tauberian theorem using (3.20) yields that

$$(3.25) \qquad\qquad M^{(2n+1)}(x) \leq O(x^n).$$

This suffices for the odd moments.

For the $(2n+2)$th moment, the binomial expansion yields

$$(3.26) \qquad M^{(2n+2)}(x+1) = \frac{1}{x} \int_0^x \sum_{\ell=1}^{n} \binom{2n+2}{2\ell} M^{(2\ell)}(t) M^{(2n+2-2\ell)}(x-t)\, dt$$

$$+ \frac{2}{x} \int_0^x M^{(2n+2)}(t)\, dt + q(x),$$

where $q(x)$ is the sum of terms which are multiples of convolutions of odd moments, and are of lower order than the first term on the right side of (3.22) by induction

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Multiplying by x and taking Laplace transforms one obtains an equation with solution

$$(3.27) \quad L^{(2n+2)}(s) = \left[\exp\left(2\int_s^\infty \frac{e^{-u}}{u} \, du\right)\right] \int_s^\infty \left[\exp\left(-2\int_v^\infty \frac{e^{-u}}{u} \, du\right)\right]$$

$$\times \left[\sum_{\ell=1}^n \binom{2n+2}{2\ell} R^{(2\ell)}(v) R^{(2n+2-2\ell)}(v) + t(v)\right] dv$$

where $t(v)$ is the Laplace transform of $q(x)$.

Let

$$(3.28) \quad f(s) = \int_s^\infty \left[\exp\left(-2\int_v^\infty \frac{e^{-u}}{u} \, du\right)\right] \left[\sum_{\ell=1}^n \binom{2n+2}{2\ell} R^{(2\ell)}(v) R^{(2n+2-2\ell)}(v) + t(v)\right] dv.$$

Then by (3.12), L'Hospital's rule, and the induction hypothesis,

$$(3.29) \quad \lim_{s \downarrow 0} s^n f(x) = \lim_{s \downarrow 0} \frac{f'(s)}{\left(\frac{1}{s^n}\right)'}$$

$$= \frac{1}{n} \sum_{\ell=1}^n \binom{2n+2}{2\ell} \frac{(2\ell)! \, \alpha^{2\ell}}{2^\ell} \cdot \frac{(2n+2-2\ell)! \, \alpha^{2n+2-2\ell}}{2^{n+1-\ell}} \, e^{2\gamma}.$$

Hence, by (3.12), (3.29),

$$(3.30) \quad \lim_{s \downarrow 0} s^{n+2} L^{(2n+2)}(s) = \frac{1}{n} \sum_{\ell=1}^n \binom{2n+2}{2\ell} \frac{(2\ell)! \, \alpha^{2\ell}}{2^\ell} \cdot \frac{(2n+2-2\ell)! \, \alpha^{2n+2-2\ell}}{2^{n+1-\ell}} \, .$$

By a Tauberian theorem,

$$(3.31) \quad \lim_{x \to \infty} x^{-(n+1)} M^{(2n+2)}(x) = \frac{1}{(n+1)! \, n} \sum_{\ell=1}^n \binom{2n+2}{2\ell} \frac{(2\ell)! \, \alpha^{2\ell} (2n+2-2\ell)! \, \alpha^{2n+2-2\ell}}{2^\ell \cdot 2^{n+1-\ell}}$$

$$= \frac{(2n+2)! \, \alpha^{2n+2}}{2^{n+1} (n+1)!}$$

and the theorem is proved.

It is clear by the nature of the computations that the moment result holds for the three models with either the Renyi or Solomon parking mechanism.

Theorem **2**. For any one-dimensional model considered, for either the Renyi or Solomon parking mechanisms, where c, α are specific to each case,

$$(3.22) \quad \lim_{x \to \infty} E\left[\exp\left(-u\left\{\frac{X(x)-\ell(x)}{\sqrt{Var\ X(x)}}\right\}\right)\right] \quad \lim_{x \to \infty} E\left[\exp\left(-u\left\{\frac{X(x)-cx}{\alpha\sqrt{x}}\right\}\right)\right] = e^{-u^2/2} \ .$$

Proof. The limiting moments of Theorem 1 are those of a $N(0,\alpha^2)$ random variable, which uniquely determine the normal law.

4. Two Dimensions.

Let

(4.1) $R(x,y)$ = total number of unit squares which may be parked on an $x \times y$ rectangle in the Renyi model in two dimensions.

Lemma.   For $x,y \to \infty$, all $n \geq 1$

$$(4.2)(i) \quad E(X(x)X(y) - \ell(x)\ell(y))^{2n} \sim \left(\frac{(2n)!}{n!2^n}\right)(c\alpha)^{2n}(xy)^n(x+y)^n$$

$$(4.2)(ii) \quad E(X(x)Y(y) - \ell(x)\ell(y))^{2n+1} \sim o((xy)(x+y))^{n+\frac{1}{2}}$$

with $Y(y)$ I.I.D. as $X(y)$.

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Proof. For x,y large, all n,

(4.3) $E[Y(y)(X(x) - \ell(x)) + \ell(x)(Y(y) - \ell(y))]^n.$

The independence of $X(x), Y(y)$, the asymptotic expressions

(4.4)(i) $\ell(x) \sim cx$

(4.4)(ii) $E(Y(y))^n = E[(Y(y) - \ell(y)) + \ell(y)]^n \sim (\ell(y))^n \sim c^n y^n.$

a binomial expansion of (4.3), Theorem 1,   an explicit asymptotic
computation of (4.2)(i) for $n=1,2$, (4.2)(ii) for $n=1$, and an induction
establish the lemma.

(4.5) $\lim\limits_{x,y \to \infty} E\left\{ \exp\left[-u\left(\dfrac{R(x,y) - \ell(x)\ell(y)}{\sqrt{E(R(x,y) - \ell(x)\ell(y))^2}} \right) \right] \right\}$

$= \lim\limits_{x,y \to \infty} E\left\{ \exp\left[-u\left(\dfrac{R(x,y) - c^2 xy}{\sqrt{c^2 \alpha^2 xy(x+y)}} \right) \right] \right\} = e^{-u^2/2}.$

Proof. For $x,y \to \infty$, $n \geq 1$,

(4.6) $E(R(x,y) - \ell(x)\ell(y))^n \cong E[(Y(y)(X(x) - \ell(x)) + (\ell(x)(Y(y) - \ell(y)))]^n.$

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The lemma immediately yields that for $x, y \to \infty$, and (4.6), that

(4.7)(i)
$$\frac{E(R(x,y) - \ell(x)\ell(y))^{2n}}{(c\alpha)^{2n}[xy(x+y)]^n} \to \frac{(2n)!}{2^n n!} \qquad \text{all} \quad n \geq 1,$$

(4.7)(ii)
$$\frac{E(R(x,y) - \ell(x)\ell(y))^{2n+1}}{[xy(x+y)]^{n+\frac{1}{2}}} \to 0, \qquad \text{all} \quad n \geq 0.$$

The right sides of (4.7)(i), (4.7)(ii) are the $2n^{th}$ and $(2n+1)^{st}$ moments of a $N(0,1)$ random variable, which determine the distribution, proving the theorem.

Similar results hold for the Solomon model and other models considered earlier. A bivariate central limit theorem holds for alternating squares or rectangles models.

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CENTRAL LIMIT THEOREM FOR

PARKING MODELS ON THE LINE AND PLANE

Asymptotic second moments for various sequential square packing models in the plane including alternating size squares, random size squares, and squares packed on an abacus grid are given or indicated for the Renyi and Solomon models. Preliminary corresponding results are given for the line. A central limit theorem follows for each model on the line and plane by computation of higher moments.

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